ANALYTICAL CONSTRUCTION TECHNIQUES FOR SOME SOLUTIONS OF FORCED PIECEWISE CONSTANT DELAY EQUATIONS

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Abstract: This paper studies a model of delayed bang-bang control under periodic saw-tooth forcing. Solutions with the same period as the forcing are constructed analytically. A simple two-parameter diagram showing the domain of existence of such solutions is derived.

Keywords: time delay, periodic motion, dynamics, feedback control

1. INTRODUCTION

This paper is concerned with analytical construction techniques for the negative feedback delay differential equation (DDE),

$$\dot{x}(t) = \operatorname{sign}(f(t) - x(t-1)), \qquad (1)$$

which can be viewed as a simple model of delayed bang-bang control. This is the limiting case of a delayed linear feedback controller with saturation as the width of the linear response region is shrunk to zero. Some analytical results for this more general delayed linear feedback controller are given in (Norbury and Wilson, 2000) and by considering (1) as a special case of this work it is possible to extend the results previously obtained. The nonlinearity in (1) is also seen in more complex relay control laws such as those studied in (Fridman *et al.*, 2002; Sieber, 2004).

Throughout this paper the forcing f is assumed to be periodic. Sharp conditions are established for the existence of solutions whose period is the same as that of the forcing.

Provided $|\dot{f}(t)| < 1$ for all t, there is a transformation between (1) and the equation

$$\dot{x}(t) = -\operatorname{sign}(x(t-1)) + g(t), \quad |g(t)| < 1, \qquad (2)$$

analysed by Fridman *et al.* (2002, 2000), which has a binary observed quantity rather than a binary output. The methods developed here do not rely on the derivative of f being bounded, and in fact the techniques are illustrated using a discontinuous saw-tooth profile for f.

Observe that the solutions of (1) are piecewise linear in t. Consequently, to construct solutions one needs only to find an initial value and the times of the subsequent minima/maxima where the gradient changes from/to ± 1.

1.1 Review of results on the unforced model

Eq. (1) with $f \equiv 0$ has an infinite family of periodic solutions of the form

$$x(t) = \begin{cases} t - T/4 & 0 \le t < T/2 \\ -t + 3T/4 & T/2 \le t < T, \end{cases} \qquad (3)$$

up to phase translation. Here the period is given by $T = 4/(1+4n)$ where $n = 0, 1, 2, \ldots$. The solution for $n = 0$ ($T = 4$) is stable; the 'fast' solutions for $n > 0$ ($T < 1$) are unstable (Fridman *et al.*, 2002).

1.2 Overview of forced analysis

Periodic forcing in the form

$$f(t) := AF(t/T), \qquad (4)$$

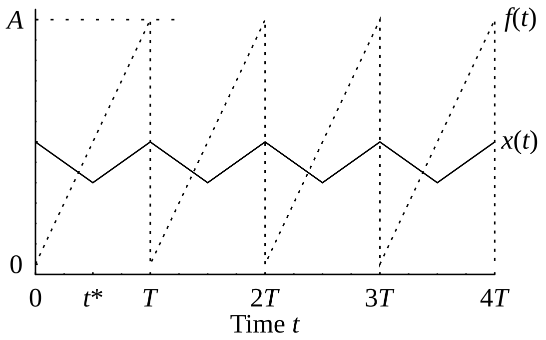

Fig. 1. A period-T solution of Eq. (1) under period-T saw-tooth forcing. There is one minimum (at $t = t^* = T/2$) and one maximum (at $t = T$) per period.

is considered, where A, $T > 0$ and F is a (fixed) period-one function. The goal is to construct period-T solutions $x(t)$ of Eq. (1), and examine the domain of existence of such solutions in the (T,A) parameter plane.

To make the details of construction as simple as possible for illustration purposes, attention is restricted to the saw-tooth profile

$$F(t) = t, \quad 0 \le t < 1, \tag{5}$$

with period-one extension. (The method for more general F is outlined in (Barton, 2003).)

Further, the only solutions considered here have exactly one minimum and one maximum per period and consequently consist of two linear (in t) segments per period, each of duration $T/2$; see Fig. 1. It follows that there exists a t^* such that:

- at $t = t^*$ $(+nT)$, $f(t) - x(t-1)$ changes from negative to positive, so that $\dot{x}(t)$ changes from -1 to $+1$ (local minimum);
- at $t = t^* + T/2$ $(+nT)$, $f(t) - x(t-1)$ changes from positive to negative, so that $\dot{x}(t)$ changes from $+1$ to -1 (local maximum);
- and there are no other points in the period where $f(t) - x(t-1)$ changes sign.

To construct solutions it is thus sufficient to find such t^* and $x(t^*)$, and having done so, to perform a back-check to ensure that the third condition is satisfied.

Given a solution $x(t)$ of (1) and a value of time t, one defines the solution history $x_t : [-1,0] \to \mathbb{R}$ by $x_t(\phi) = x(t+\phi)$ (see Hale and Lunel (1993)). This may be used to split the analysis of period-T solutions into two cases:

- *Long period solutions* with $T \ge 2$ (the easier case). The time $T/2$ between consecutive turning points equals or exceeds the delay time of 1. This introduces a 'loss of memory' effect where the solution history at turning points is wholly linear in t. For example, at the minimum at $t = t^*$, the history takes the form

$$x_{t^*}(\phi) = x(t^*) - \phi. \tag{6}$$

  It follows that $x(t^*-1) = x(t^*) + 1$ and the condition for the minimum is that $f(t) - x(t) - 1$ goes from negative to positive at $t = t^*$.
- *Short period solutions* with $T < 2$ (the more difficult case). The time $T/2$ between turning points is less than the delay time of 1, and the solution history consists of multiple linear segments. In this case, the solution history cannot be determined by the current value of $x(t)$ alone. Instead, knowledge of the locations of previous turning points is required to compute the next turning point.

Long period solutions are considered in Section 3 and short period solutions are considered in Section 4.

2. COARSE BOUNDS ON THE EXISTENCE OF PERIOD-T SOLUTIONS

Here it is shown that when the forcing amplitude A is sufficiently large, there exists a period-T solution of the required form, i.e., with exactly one minimum and one maximum per period.

Take $A > T$, so that $\dot{f} > 1$ and $\dot{f}(t) - \dot{x}(t-1) > 0$ where f is differentiable (since $\dot{x} = \pm 1$). It follows that $f(t) - x(t-1)$ is increasing for $t \in (0,T)$. Hence it can only pass through zero in a negative-to-positive direction, i.e., the local maximum cannot occur in $(0,T)$ and must be located at the discontinuity of f at $t = nT$. Consequently, the local minimum is given by $t^* = T/2$ $(+nT)$. It remains to find $x(t^*)$ such that $f(t) - x(t-1)$ increases through zero at $t = t^* = T/2$, that is

$$x(T/2-1) = f(T/2) = A/2. \tag{7}$$

Finally one needs to back-check that the jump in f is sufficient for $f(t) - x(t-1)$ to decrease through zero at $t = T$:

$$0 \le x(T-1) \le A. \tag{8}$$

Since $x(t)$ has gradient ± 1, Eq. (7) yields $x(T-1) \in [A/2 - T/2, A/2 + T/2]$, which is satisfied simultaneously with (8) if $A > T$. This is thus the coarse bound for the existence of such a solution.

The challenge in the next two sections is to derive sharp conditions for the existence of such solutions in the (T,A) plane as A is reduced.

3. LONG PERIOD SOLUTIONS

In this section the long period case $T \ge 2$ is considered and, as in the previous section, solutions are constructed with a local maximum at the discontinuity $t = nT$ of f. However, the requirement $A > T$ is now relaxed.

As before, it is required that $f(t) - x(t-1)$ increases through zero at $t = T/2$ and that the jump in f is sufficiently strong to force $f(t) - x(t-1)$ to change from

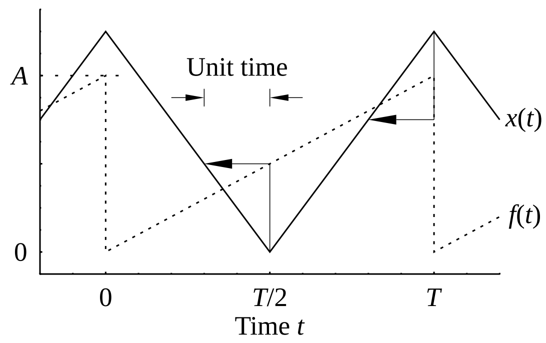

Fig. 2. A solution of equation (1) under long period ($T \geq 2$) forcing. The arrows indicate the point $x(t-1)$ against which $f(t)$ is compared. For the minima/maxima of $x(t)$ to exist as shown, it is required that $x(T/2-1) = A/2$ and $0 \leq x(T-1) \leq A$.

positive to negative at $t = T$. Hence again Eqs. (7) and (8) must hold; see Fig. 2.

The 'loss of memory' effect in the long period solutions allows one to write $x(t-1)$ explicitly in terms of $x(t)$ at the turning points, and this gives

$$x(T/2-1) = x(T/2)+1, \tag{9}$$
$$x(T-1) = x(T)-1. \tag{10}$$

Since the solution is linear in t between turning points, this produces $x(T) = x(T/2)+T/2$, and combining with (7), (9) and (10) thus gives

$$x(T-1) = A/2+T/2-2. \tag{11}$$

Combining with (8), the sharp bound for the existence of a period-T solution of the required form is found to be

$$A \geq |T-4|. \tag{12}$$

When $A = |T-4|$ one may also find other period-T solutions with one minimum and one maximum per period, where the maximum does not lie on the discontinuity of f.

4. SHORT PERIOD SOLUTIONS

In this section the short period case $T < 2$ is considered. As before, solutions are constructed consisting of two linear segments per period, with the local maximum at the discontinuity $t = nT$ of f, and the local minimum at $t = T/2$ $(+nT)$. As before, Eqs. (7) and (8) must hold, and one needs to perform a back-check to ensure that the constructed $f(t)-x(t-1)$ only changes sign at $t = T/2$ $(+nT)$ and $t = T$ $(+nT)$. As was remarked earlier, the solution history $x_t(\phi)$ is no longer linear in ϕ in the short period case, and hence the details of the back-check are more complicated than before.

Without loss of generality, a linear segment of the solution $x(t) = x(T/2)+(t-T/2)$, $t \in [T/2,T]$ is considered here, running from the local minimum to the local maximum; see Fig. 3. (The analysis of the segment running from the local maximum to the local minimum progresses similarly.) Further, consider the piecewise linear solution segment $x(t-1)$, $t \in [T/2,T]$. Depending on the precise value of T, the back-check analysis splits into two sub-cases:

(i) $x(t-1)$ has a *local maximum* for $t \in [T/2,T]$ as shown in Fig. 3(a). Then $f(t)$ and $x(t-1)$ can only gain extra intersections as the gradient of f decreases through $+1$, which is equivalent to reducing A through T. Hence the back-check produces the sharp (necessary) constraint $A > T$.

(ii) $x(t-1)$ has a *local minimum* for $t \in [T/2,T]$ as shown in Fig. 3(b). As the amplitude, and consequently the gradient of f is reduced, it is not possible to generate further intersections of $f(t)$ and $x(t-1)$. Consequently, the back-check is automatically satisfied.

In case (i), the necessary condition $A > T$ is identical to the sufficient condition derived in Section 2. Hence, $A > T$ is a necessary *and* sufficient condition for the existence of solutions.

Observe from geometrical considerations that if r is defined by

$$1 = kT + r \quad \text{where} \quad k \in \mathbb{Z}, \quad 0 \leq r < T, \tag{13}$$

(i.e., r is the remainder when the delay is divided by the period) then

- if $r > T/2$, case (i) holds; and
- if $0 \leq r \leq T/2$, case (ii) holds.

For case (ii) one needs to consider what other necessary constraints are introduced by Eqs. (7) and (8). Recall from Eq. (7) that $x(T/2-1) = A/2$, and consequently

$$\begin{aligned} x(T-1) &= x(T/2-1) - r + (T/2-r), \\ &= A/2+T/2-2r, \end{aligned} \tag{14}$$

since the solution has gradient -1 for time r, and gradient $+1$ for time $T/2-r$, between $t = T/2-1$ and $t = T-1$.

For a maximum of $x(t)$ to occur at the discontinuity $t = T$ of f, Eq. (8) must hold, which yields

$$A \geq T-4r \quad \text{and} \quad A \geq 4r-T, \tag{15}$$

when combined with (14).

5. THE (T,A)-PLANE

The results of Sections 3 and 4 may be combined to give a two-parameter diagram in the (T,A)-plane describing the existence of period-T solutions of Eqs. (1), (4), and (5), with one minimum and one maximum per period; see Fig. 4. The key features are as follows:

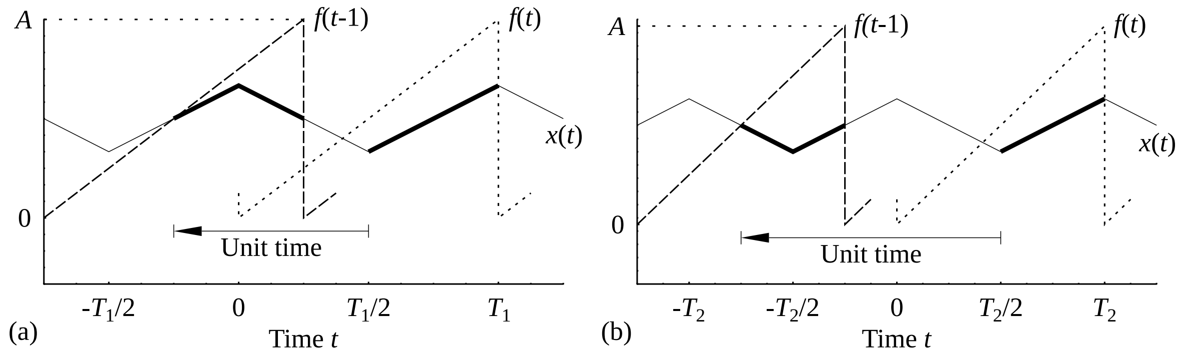

Fig. 3. Solutions of (1) under short period forcing ($T < 2$). The bold segments indicate $x(t)$ and $x(t-1)$ for $t \in [T/2, T]$. Panels (a) and (b) show the cases where the $x(t-1)$ segment contains a local maximum and a local minimum respectively.

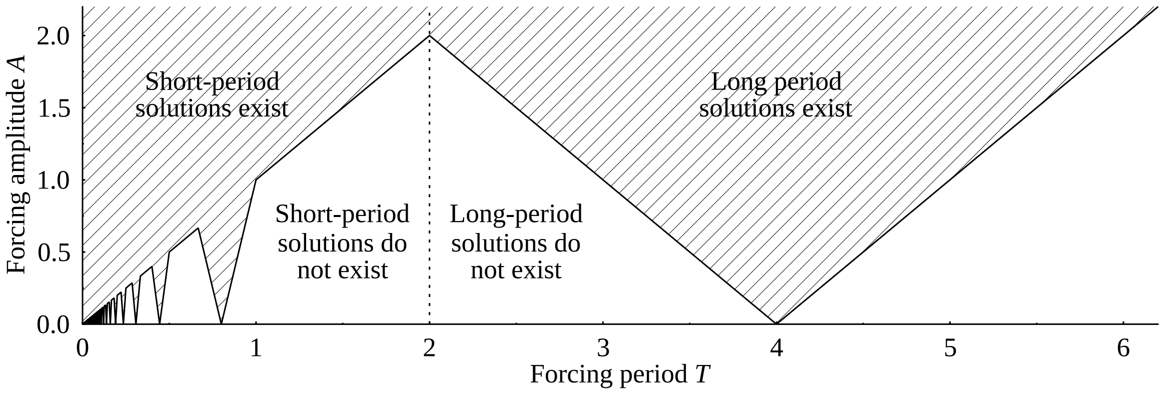

Fig. 4. The shaded region of the figure indicates the parameter values for which there exists a period-T solution of Eqs. (1), (4), and (5) with one minimum and one maximum per period. The tongue-like grooves are centred on those periods T for which there is a solution (3) to the unforced equation.

For $T \geq 2$,

- the existence boundary is given by $A = |T - 4|$.

For $T < 2$, either

- $r > T/2$ (see (13)) in which case the boundary of existence is a 'cap' $A = T$, or
- $r \leq T/2$ in which case the boundary of existence is a 'groove' given by (15).

There are infinitely many 'grooves', each centred on $T = 4/(1+4n)$, $n = 1, 2, \ldots$, and at these values of T, the forced solutions deform continuously on to the unforced 'fast' solutions (see (3)) as $A \to 0$. Away from these grooves, numerical simulations indicate that there is typically a jump to quasi-periodicity as the boundary of existence is crossed.

6. CONCLUSION

In this paper, solutions to Eq. (1) under saw-tooth forcing have been constructed. These solutions have the same period as the forcing, and can be divided into two types: long-period solutions and short-period solutions. Construction of long-period solutions ($T \geq 2$) is less problematic than the short-period solutions ($T < 2$) due to the 'loss of memory' effect present. The construction of short-period solutions shows the existence of infinitely many solutions which, as the forcing amplitude is decreased, deform continuously on to the unforced 'fast' solutions.

Simulations indicate that the solutions that were constructed in this paper are at least locally stable, and work on proofs is currently under way. When $T \neq 4/(1+4n)$, there is the possibility of very complicated dynamics as A is reduced and the existence boundary is crossed. Currently this behaviour is being investigated using a combination of further analytical construction techniques and numerical continuation with nearby smoothed-off models, using the package DDE-BIFTOOL (Engelborghs *et al.*, 2001). These results will be reported elsewhere.

7. ACKNOWLEDGEMENTS

The authors would like to thank Bernd Krauskopf for helpful comments and suggestions during the writing of this paper.

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